Poisson–Hopf limit of quantum algebras

2009 J. Phys. A: Math. Theor. 42 275202

(http://iopscience.iop.org/1751-8121/42/27/275202)

IOP Publishing   Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **42** (2009) 275202 (9pp)   doi:10.1088/1751-8113/42/27/275202

Poisson–Hopf limit of quantum algebras

A Ballesteros[1], E Celeghini[2] and M A del Olmo[3]

[1] Departamento de Física, Universidad de Burgos, E-09006, Burgos, Spain
[2] Departimento di Fisica, Università di Firenze and INFN–Sezione di Firenze I50019 Sesto Fiorentino, Firenze, Italy
[3] Departamento de Física Teórica, Universidad de Valladolid, E-47005, Valladolid, Spain.

E-mail: angelb@ubu.es, celeghini@fi.infn.it and olmo@fta.uva.es

Received 12 March 2009, in final form 14 May 2009
Published 11 June 2009
Online at stacks.iop.org/JPhysA/42/275202

Abstract
The Poisson–Hopf analogue of an arbitrary quantum algebra $U_z(g)$ is constructed by introducing a one-parameter family of quantizations $U_{z,\hbar}(g)$ depending explicitly on $\hbar$ and by taking the appropriate $\hbar \to 0$ limit. The q-Poisson analogues of the $su(2)$ algebra are discussed and the novel $su_q^{\mathcal{P}}(3)$ case is introduced. The q–Serre relations are also extended to the Poisson limit. This approach opens the perspective for possible applications of higher rank q-deformed Hopf algebras in semiclassical contexts.

PACS numbers: 02.20.Uw, 03.65.Sq
Mathematics Subject Classification: 17B63, 17B37, 81R50

1. Introduction

Quantum groups were initially introduced as quantizations of Poisson–Lie groups associated with certain solutions of the classical Yang–Baxter equation. In this context, the deformation parameters were taken as $q = e^z$, where z is the constant that governs the noncommutativity of the algebra of observables given by the quantum group entries, and quantum algebras were obtained as the Hopf algebra dual of quantum groups (for a detailed discussion, see [1–5] and references therein). In the case of the transition from classical to quantum physical models, the deformation parameter z was interpreted as the Planck constant $\hbar$.

However, in more general contexts z is a parameter whose geometric/physical meaning has to be elucidated for each particular case. In fact, quantum groups and quantum algebras were soon considered as 'abstract' Hopf algebras (being both noncommutative and non-co-commutative) in order to explore whether these new objects can be considered as new symmetries of some physically relevant systems. The keystone of this approach was the discovery of the $su_q(2)$ invariance of the Heisenberg spin XXZ chain [6, 7], that was followed by a number of results exploiting quantum algebra symmetries in two-dimensional models [8]. Indeed, in the XXZ chain the 'quantum' deformation parameter q is clearly identified with the anisotropy of the chain, which is completely independent of the (truly quantum) $\hbar$ constant. And the same independence with respect to $\hbar$ can be traced in many other physical applications of quantum algebras and groups, such as for instance lattice systems (where q is related to the lattice length) [9, 10], deformations of kinematical symmetries (in which the deformation parameter is a fundamental scale, see [11–15] and references therein), effective nuclear models [16, 17], etc.

The aim of this paper is to introduce a mathematical framework for quantum algebras in which both the deformation parameter $z = \log q$ and the Planck constant $\hbar$ are independently and simultaneously considered. This approach is described in section 2, where for any quantum algebra $U_z(g)$ we construct a one-parameter family of equivalent quantizations $U_{z,\hbar}(g)$ that depends explicitly on both $\hbar$ and z. Then, a q-Poisson–Hopf algebra is obtained as the $\hbar \to 0$ limit, thus getting—as in the Lie case– a proper classical-mechanical limit of quantum systems endowed with an arbitrary quantum algebra symmetry. This approach can be interesting to construct new (classical) integrable systems having a parameter z that can control their dynamical behaviour. Since the relevant structure of the deformed Poisson–Lie algebra is the coproduct, we discuss it in detail. Co-unity and antipode can be obtained in the same way.

In section 3 the method is illustrated by applying it to the quantum deformations of $su(2)$, considering both the standard and the non-standard ones. In this elementary case the q-Poisson–Hopf algebras obtained as the $\hbar \to 0$ limit seem to be formally identical to the original quantum algebras. The case of $su(3)$, fully described in section 4, is the first non-trivial one since by starting from the quantum algebra $U_{z,\hbar}(su(3))$ we obtain a q-Poisson $su(3)$ algebra (that we shall call $su_q^{\mathcal{P}}(3)$) which is quite different from the former. To the best of our knowledge, this is a new q-Poisson–Hopf algebra that could be used, for instance, in order to construct integrable deformations of higher rank classical Gaudin models [18–21] by using the approach presented in [22] or to consider the semiclassical limit of the $U_z(su(3))$ dynamics from the viewpoint of [23]. With this in mind, the explicit form of the two Casimir functions for $su_q^{\mathcal{P}}(3)$ is explicitly found. A concluding section closes the paper, in which the Poisson analogue of the q-Serre relations for higher rank q-Poisson-Hopf algebras is consistently defined.

2. Quantum algebras and Poisson–Hopf limit

Let us recall that a Lie bialgebra (g, δ) is a Lie algebra g

$$[X_i, X_j] = f_{ij}^k X_k \tag{1}$$

together with a co-commutator map $\delta : g \to g \otimes g$ given by

$$\delta(X_i) = c_i^{jk} X_j \otimes X_k \tag{2}$$

such that c_i^{jk} defines a (dual) Lie algebra and fulfils the appropriate compatibility condition (see [4, 5] for details).

Given an arbitrary Lie bi-algebra (g, δ) the quantum algebra $(U_z(g), \Delta_z)$ (that is obtained through the analytic procedure described in [24, 25]) is the Hopf algebra deformation $(U_z(g), \Delta_z)$ of the universal enveloping algebra of g, $U(g)$, compatible with the deformed coproduct $\Delta_z(X)$ whose leading order terms are

$$\Delta_z(X) = \Delta_0(X) + z\delta(X) + o[z^2].$$

Let us now consider the one-parameter family of equivalent Lie bi-algebras $(g_\hbar, \delta)$ defined by (2) and

$$[X_i, X_j] = \hbar f_{ij}^k X_k. \tag{3}$$

Note that when $\hbar = 1$ we recover the original commutation relations (1). If we quantize $(g_\hbar, \delta)$ by using the method described in [25] we obtain the two-parameter quantum algebra $(U_{z,\hbar}(g), \Delta_{z,\hbar})$, that depends explicitly on $\hbar$. Then, the Poisson limit ($\hbar \to 0$) of $(U_{z,\hbar}(g), \Delta_{z,\hbar})$ can uniquely be defined and gives the Poisson–Hopf algebra $(Fun(g_z), \Delta_z^{\mathcal{P}})$. This q-Poisson algebra is just a Poisson–Lie structure on the group g_z whose Lie algebra is determined by the dual δ^* of the Lie bialgebra map (2), i.e., by the structure tensor c_i^{jk} (see [4, 5]). The Poisson bracket on $\mathrm{Fun}(g_z)$ is given by

$$\{X, Y\} := \lim_{\hbar\to 0} \frac{[X, Y]}{\hbar} \tag{4}$$

and the coproduct map

$$\Delta_z^{\mathcal{P}}(X) := \lim_{\hbar\to 0} \Delta_{z,\hbar}(X) \tag{5}$$

is a Poisson algebra homomorphism between $\mathrm{Fun}(g_z)$ and $\mathrm{Fun}(g_z) \otimes \mathrm{Fun}(g_z)$.

If we deal with non-deformed Lie algebras, its coproduct is the primitive one and the commutation rules are linear: thus the Poisson limit (4) and (5) leads to the same formal structure where commutation rules have been just replaced by Poisson brackets. In contrast, quantum algebras introduce nonlinear functions of the generators both at the level of the commutation rules and of the coproduct. This implies that the q-Poisson structure given by the limit $\hbar \to 0$ can be formally different to the original quantum algebra structure. In fact, such limit allows us to remove contributions in the deformation that arise as reordering terms, as shown in the following sections.

Summarizing, in this paper we introduce the following commutative diagram

$$\begin{tikzcd}
(U(g), \Delta_0) & (U_\hbar(g), \Delta_0) \arrow[l, "\hbar=1"'] \arrow[r, "\hbar\to 0"] & (\mathrm{Fun}(g), \Delta_0^{\mathcal{P}}) \\
(U_z(g), \Delta_z) \arrow[u, "z\to 0"'] & (U_{z,\hbar}(g), \Delta_{z,\hbar}) \arrow[l, "\hbar=1"'] \arrow[r, "\hbar\to 0"] \arrow[u, "z\to 0"'] & (\mathrm{Fun}(g_z), \Delta_z^{\mathcal{P}}) \arrow[u, "z\to 0"']
\end{tikzcd}$$

where we focus on the lower right corner, $(\mathrm{Fun}(g_z), \Delta_z^{\mathcal{P}})$, that we define in a constructive way by starting from any Lie bi-algebra. This general approach is illustrated in the following sections through the $su_q(2)$ and $su_q(3)$ examples. In particular, the q-Poisson algebra presented in section 4 is, to the best of our knowledge, the first example of a Hopf algebra deformation of the Poisson $su(3)$ algebra.

3. q-Poisson–Hopf algebras related to $su(2)$

In the $su(2)$ case, two well-known quantum deformations do exist: the standard one [26, 27] and the non-standard (or Jordanian) deformation [28]. As we shall see in what follows, their corresponding q-Poisson–Hopf algebras are formally equivalent to the quantum algebras from which they have been obtained.

3.1. Standard q-Poisson algebra $su_q^{\mathcal{P}}(2)$

The $su(2)$ commutation rules are

$$[F_{12}, F_{21}] = 2H \qquad [H, F_{12}] = F_{12} \qquad [H, F_{21}] = -F_{21}, \tag{6}$$

where $H = (H_1 - H_2)/2$. The standard $su(2)$ Lie bialgebra is given by

$$\delta(H) = 0 \qquad \delta(F_{12}) = H \wedge F_{12} \qquad \delta(F_{21}) = H \wedge F_{21}. \tag{7}$$

The well-known quantum algebra deformation of (6) and (7) reads

$$[F_{12}, F_{21}] = \frac{\sinh(2zH)}{z} \qquad [H, F_{12}] = F_{12} \qquad [H, F_{21}] = -F_{21} \tag{8}$$

$$\begin{aligned} \Delta_z(H) &= H \otimes 1 + 1 \otimes H \\ \Delta_z(F_{12}) &= \mathrm{e}^{zH} \otimes F_{12} + F_{12} \otimes \mathrm{e}^{-zH} \\ \Delta_z(F_{21}) &= \mathrm{e}^{zH} \otimes F_{21} + F_{21} \otimes \mathrm{e}^{-zH} \end{aligned} \tag{9}$$

Now if we consider the $\hbar$-parameter Lie algebra

$$[F_{12}, F_{21}] = 2\hbar H \qquad [H, F_{12}] = \hbar F_{12} \qquad [H, F_{21}] = -\hbar F_{21} \tag{10}$$

we would obtain a quantization in which the coproduct (9) does not formally change, but we have

$$[F_{12}, F_{21}] = \hbar \frac{\sinh(2zH)}{z} \qquad [H, F_{12}] = \hbar F_{12} \qquad [H, F_{21}] = -\hbar F_{21}. \tag{11}$$

As a consequence, the $\hbar$-deformed Casimir operator is shown to be

$$\begin{aligned} C_q &= \frac{\sinh^2(zH)}{z^2} \cosh(z\hbar) + \frac{1}{2}[F_{12}, F_{21}]_+ \\ &= \frac{\sinh(zH)}{z} \frac{\sinh z(H + \hbar)}{z} + F_{21}F_{12} \\ &= \frac{\sinh(zH)}{z} \frac{\sinh z(H - \hbar)}{z} + F_{12}F_{21}, \end{aligned} \tag{12}$$

where the role of $\hbar$ can be easily appreciated.

We stress that if we perform the following substitution

$$H \to \hbar H \qquad z \to z/\hbar \qquad F_{12} \to \hbar F_{12} \qquad F_{21} \to \hbar F_{21}$$

the parameter $\hbar$ can be reabsorbed and we get just the standard expressions (8) and (9), thus confirming that the quantum algebra $U_z(su(2))$ depends on only one essential parameter. However, since the limit $\hbar \to 0$ does not commute with this mapping, the $\hbar$ parameter is essential in order to define the proper q-Poisson algebra.

As a result, the q-Poisson algebra $su_q^{\mathcal{P}}(2) \equiv \left(\mathrm{Fun}(g_z), \Delta_z^{\mathcal{P}}\right)$ is obtained as the commutative algebra of functions endowed with the Poisson bracket and coproduct map coming, respectively, from the limits (4) and (5). Namely,

$$\{F_{12}, F_{21}\} = \frac{\sinh(2zH)}{z} \qquad \{H, F_{12}\} = F_{12} \qquad \{H, F_{21}\} = -F_{21} \tag{13}$$

$$\begin{aligned} \Delta_z^{\mathcal{P}}(H) &= H \otimes 1 + 1 \otimes H \\ \Delta_z^{\mathcal{P}}(F_{12}) &= \mathrm{e}^{zH} \otimes F_{12} + F_{12} \otimes \mathrm{e}^{-zH} \\ \Delta_z^{\mathcal{P}}(F_{21}) &= \mathrm{e}^{zH} \otimes F_{21} + F_{21} \otimes \mathrm{e}^{-zH} \end{aligned} \tag{14}$$

which is indeed a Poisson–Hopf algebra, as it can be checked by direct computation. Finally, a unique q-deformed Casimir function is obtained from (12) as

$$C_q^{\mathcal{P}} = \lim_{\hbar \to 0} C_q = \frac{1}{z^2} \sinh^2(zH) + F_{12}F_{21}.$$

3.2. Non-standard q-Poisson $su^{\mathcal{P}}_{ns}(2)$

The same procedure can be applied to the non-standard deformation of $su(2)$. Explicitly, the non-standard $\hbar$-parameter Lie bialgebra would be given by (10) and

$$\delta(F_{12}) = 0, \qquad \delta(H) = H \wedge F_{12}, \qquad \delta(F_{21}) = F_{21} \wedge F_{12}. \tag{15}$$

The corresponding quantum algebra is [28]

$$\begin{aligned}
&[H, F_{12}] = \hbar \frac{\sinh(zF_{12})}{z} \qquad\qquad [F_{12}, F_{21}] = 2\hbar H \\
&[H, F_{21}] = -\frac{\hbar}{2}\left(F_{21}\cosh(zF_{12}) + \cosh(zF_{12})F_{21}\right) \\
&\Delta_{z,\hbar}(F_{12}) = 1 \otimes F_{12} + F_{12} \otimes 1 \\
&\Delta_{z,\hbar}(H) = \mathrm{e}^{-zF_{12}} \otimes H + H \otimes \mathrm{e}^{zF_{12}} \\
&\Delta_{z,\hbar}(F_{21}) = \mathrm{e}^{-zF_{12}} \otimes F_{21} + F_{21} \otimes \mathrm{e}^{zF_{12}}
\end{aligned}$$

whose Casimir operator reads

$$\mathcal{C}_{\mathrm{ns}} = H^2 + \frac{1}{2}\left(\frac{\sinh zF_{12}}{z} F_{21} + F_{21} \frac{\sinh zF_{12}}{z}\right) + \frac{\hbar^2}{4}\cosh^2(zF_{12}). \tag{16}$$

Now, by performing the same $\hbar \to 0$ limit given by (4) and (5) we get the non-standard q-Poisson–Hopf algebra $su^{\mathcal{P}}_{\mathrm{ns}}(2)$

$$\begin{aligned}
&\{H, F_{12}\} = \frac{\sinh(zF_{12})}{z} \qquad\qquad \{F_{12}, F_{21}\} = 2H \\
&\{H, F_{21}\} = -F_{21}\cosh(zF_{12})
\end{aligned} \tag{17}$$

$$\begin{aligned}
&\Delta^{\mathcal{P}}_z(F_{12}) = 1 \otimes F_{12} + F_{12} \otimes 1 \\
&\Delta^{\mathcal{P}}_z(H) = \mathrm{e}^{-zF_{12}} \otimes H + H \otimes \mathrm{e}^{zF_{12}} \\
&\Delta^{\mathcal{P}}_z(F_{21}) = \mathrm{e}^{-zF_{12}} \otimes F_{21} + F_{21} \otimes \mathrm{e}^{zF_{12}}.
\end{aligned} \tag{18}$$

And the Casimir function is

$$\mathcal{C}^{\mathcal{P}}_{\mathrm{ns}} = H^2 + \frac{\sinh(zF_{12})}{z} F_{21}. \tag{19}$$

4. The q-Poisson $su^{\mathcal{P}}_q(3)$ algebra

In this case we will refer to [29, 30], where the standard deformation of $su(3)$ is obtained by starting from the Weyl–Drinfeld basis of the bi-algebra where all roots are well defined. In this way a complete description of the whole structure for $u_q(3) \equiv su_q(3) \oplus u(1)$, real form of $A^q_2 \oplus A_1$, is obtained. In this basis, the explicit commutation rules for $su(3)$ are $(i, j, k = 1, 2, 3)$:

$$\begin{aligned}
&[H_i, H_j] = 0, \\
&[H_i, F_{jk}] = \hbar(\delta_{ij} - \delta_{ik})F_{jk}, \\
&[F_{ij}, F_{kl}] = \hbar(\delta_{jk}F_{il} - \delta_{il}F_{kj}) + \hbar\delta_{jk}\delta_{il}(H_i - H_j).
\end{aligned} \tag{20}$$

The canonical Lie bialgebra structure is determined by the co-commutator:

$$\begin{aligned}
&\delta(H_i) = 0, \\
&\delta(F_{ij}) = \tfrac{1}{2}(H_i - H_j) \wedge F_{ij} + \textstyle\sum_{k=i+1}^{j-1} F_{ik} \wedge F_{kj} \qquad (i < j), \\
&\delta(F_{ij}) = \tfrac{1}{2}(H_j - H_i) \wedge F_{ij} - \textstyle\sum_{k=j+1}^{i-1} F_{ik} \wedge F_{kj} \qquad (i > j).
\end{aligned} \tag{21}$$

4.1. The quantum algebra $U_{z,\hbar}(su(3))$

By making use of the quantization approach described in [25] onto the canonical Lie bi-algebra structure (20)–(21), a long but straightforward computation gives rise to the following coproduct:

$$
\begin{aligned}
\Delta_{z,\hbar}(H_i) &= H_i \otimes 1 + 1 \otimes H_i\\
\Delta_{z,\hbar}(F_{12}) &= \mathrm{e}^{z(H_1-H_2)/2} \otimes F_{12} + F_{12} \otimes \mathrm{e}^{-z(H_1-H_2)/2}\\
\Delta_{z,\hbar}(F_{23}) &= \mathrm{e}^{z(H_2-H_3)/2} \otimes F_{23} + F_{23} \otimes \mathrm{e}^{-z(H_2-H_3)/2}\\
\Delta_{z,\hbar}(F_{21}) &= \mathrm{e}^{z(H_1-H_2)/2} \otimes F_{21} + F_{21} \otimes \mathrm{e}^{-z(H_1-H_2)/2}\\
\Delta_{z,\hbar}(F_{32}) &= \mathrm{e}^{z(H_2-H_3)/2} \otimes F_{32} + F_{32} \otimes \mathrm{e}^{-z(H_2-H_3)/2}\\
\Delta_{z,\hbar}(F_{13}) &= \mathrm{e}^{z(H_1-H_3)/2} \otimes F_{13} + F_{13} \otimes \mathrm{e}^{-z(H_1-H_3)/2}\\
&\quad + \frac{2}{\hbar}\sinh(z\hbar/2)(\mathrm{e}^{z(H_2-H_3)/2}F_{12} \otimes \mathrm{e}^{-z(H_1-H_2)/2}F_{23}\\
&\quad - \mathrm{e}^{z(H_1-H_2)/2}F_{23} \otimes \mathrm{e}^{-z(H_2-H_3)/2}F_{12})\\
\Delta_{z,\hbar}(F_{31}) &= \mathrm{e}^{z(H_1-H_3)/2} \otimes F_{31} + F_{31} \otimes \mathrm{e}^{-z(H_1-H_3)/2}\\
&\quad + \frac{2}{\hbar}\sinh(z\hbar/2)(\mathrm{e}^{z(H_2-H_3)/2}F_{21} \otimes \mathrm{e}^{-z(H_1-H_2)/2}F_{32}\\
&\quad - \mathrm{e}^{z(H_1-H_2)/2}F_{32} \otimes \mathrm{e}^{-z(H_2-H_3)/2}F_{21}).
\end{aligned}
$$

Concerning the deformed commutation rules, those in which the elements of the Cartan subalgebra are involved will remain undeformed with respect to (20). The remaining ones are found to be

$$
\begin{aligned}
&[F_{12}, F_{23}] = \hbar F_{13} && [F_{32}, F_{21}] = \hbar F_{31}\\
&[F_{12}, F_{13}] = \frac{4}{\hbar}\left(\sinh\frac{z\hbar}{2}\right)^2 F_{12}F_{23}F_{12} && [F_{13}, F_{23}] = \frac{4}{\hbar}\left(\sinh\frac{z\hbar}{2}\right)^2 F_{23}F_{12}F_{23}\\
&[F_{31}, F_{21}] = \frac{4}{\hbar}\left(\sinh\frac{z\hbar}{2}\right)^2 F_{21}F_{32}F_{21} && [F_{32}, F_{31}] = \frac{4}{\hbar}\left(\sinh\frac{z\hbar}{2}\right)^2 F_{32}F_{21}F_{32}\\
&[F_{23}, F_{21}] = 0 && [F_{12}, F_{32}] = 0\\
&[F_{12}, F_{21}] = \hbar\frac{\sinh(z(H_1-H_2))}{z} && [F_{23}, F_{32}] = \hbar\frac{\sinh(z(H_2-H_3))}{z}
\end{aligned}
$$

$$
\begin{aligned}
[F_{13}, F_{21}] &= -\frac{2}{z}\sinh\frac{z\hbar}{2}\cosh\left(z\left(H_1-H_2+\frac{\hbar}{2}\right)\right)F_{23}\\
[F_{13}, F_{32}] &= \frac{2}{z}\sinh\frac{z\hbar}{2}\cosh\left(z\left(H_2-H_3+\frac{\hbar}{2}\right)\right)F_{12}\\
[F_{12}, F_{31}] &= -\frac{2}{z}\sinh\frac{z\hbar}{2}\cosh\left(z\left(H_1-H_2-\frac{\hbar}{2}\right)\right)F_{32}\\
[F_{23}, F_{31}] &= \frac{2}{z}\sinh\frac{z\hbar}{2}\cosh\left(z\left(H_2-H_3-\frac{\hbar}{2}\right)\right)F_{21}\\
[F_{13}, F_{31}] &= \hbar\frac{\sinh(z(H_1-H_3))}{z} + \frac{2}{z\hbar}\left(\sinh\frac{z\hbar}{2}\right)^2\sinh(z(H_1-H_2))\,[F_{23}, F_{32}]_+\\
&\quad + \frac{2}{z\hbar}\left(\sinh\frac{z\hbar}{2}\right)^2\sinh(z(H_2-H_3))\,[F_{12}, F_{21}]_+
\end{aligned}
$$

4.2. The Poisson-Hopf limit

If we compute the limits (4) and (5) of the above expressions we get the following Poisson–Hopf algebra $su_q^{\mathcal{P}}(3)$ with coproduct

$$
\begin{aligned}
\Delta_z^{\mathcal{P}}(H_i) &= H_i \otimes 1 + 1 \otimes H_i, \quad i = 1, 2, 3\\
\Delta_z^{\mathcal{P}}(F_{12}) &= \mathrm{e}^{z(H_1-H_2)/2} \otimes F_{12} + F_{12} \otimes \mathrm{e}^{-z(H_1-H_2)/2}\\
\Delta_z^{\mathcal{P}}(F_{23}) &= \mathrm{e}^{z(H_2-H_3)/2} \otimes F_{23} + F_{23} \otimes \mathrm{e}^{-z(H_2-H_3)/2}\\
\Delta_z^{\mathcal{P}}(F_{21}) &= \mathrm{e}^{z(H_1-H_2)/2} \otimes F_{21} + F_{21} \otimes \mathrm{e}^{-z(H_1-H_2)/2}\\
\Delta_z^{\mathcal{P}}(F_{32}) &= \mathrm{e}^{z(H_2-H_3)/2} \otimes F_{32} + F_{32} \otimes \mathrm{e}^{-z(H_2-H_3)/2}\\
\Delta_z^{\mathcal{P}}(F_{13}) &= \mathrm{e}^{z(H_1-H_3)/2} \otimes F_{13} + F_{13} \otimes \mathrm{e}^{-z(H_1-H_3)/2} + z(\mathrm{e}^{z(H_2-H_3)/2} F_{12} \otimes \mathrm{e}^{-z(H_1-H_2)/2} F_{23}\\
&\qquad - \mathrm{e}^{z(H_1-H_2)/2} F_{23} \otimes \mathrm{e}^{-z(H_2-H_3)/2} F_{12})\\
\Delta_z^{\mathcal{P}}(F_{31}) &= \mathrm{e}^{z(H_1-H_3)/2} \otimes F_{31} + F_{31} \otimes \mathrm{e}^{-z(H_1-H_3)/2} + z(\mathrm{e}^{z(H_2-H_3)/2} F_{21} \otimes \mathrm{e}^{-z(H_1-H_2)/2} F_{32}\\
&\qquad - \mathrm{e}^{z(H_1-H_2)/2} F_{32} \otimes \mathrm{e}^{-z(H_2-H_3)/2} F_{21})
\end{aligned}
$$

The deformed Poisson brackets are

$$
\begin{aligned}
&\{H_i, H_j\} = 0, && \{H_i, F_{jk}\} = (\delta_{ij} - \delta_{ik}) F_{jk},\\
&\{F_{12}, F_{23}\} = F_{13} && \{F_{32}, F_{21}\} = F_{31}\\
&\{F_{12}, F_{13}\} = \{F_{12}, \{F_{12}, F_{23}\}\} = z^2 F_{12}^2 F_{23} && \{F_{13}, F_{23}\} = \{\{F_{12}, F_{23}\}, F_{23}\} = z^2 F_{23}^2 F_{12}\\
&\{F_{31}, F_{21}\} = \{\{F_{32}, F_{21}\}, F_{21}\} = z^2 F_{21}^2 F_{32} && \{F_{32}, F_{31}\} = \{F_{32}, \{F_{32}, F_{21}\}\} = z^2 F_{32}^2 F_{21}\\
&\{F_{23}, F_{21}\} = 0 && \{F_{12}, F_{32}\} = 0.\\
&\{F_{12}, F_{21}\} = \tfrac{1}{z} \sinh(z(H_1 - H_2)) && \{F_{23}, F_{32}\} = \tfrac{1}{z} \sinh(z(H_2 - H_3))\\
&\{F_{13}, F_{21}\} = -\cosh(z(H_1 - H_2)) F_{23} && \{F_{13}, F_{32}\} = \cosh(z(H_2 - H_3)) F_{12}\\
&\{F_{12}, F_{31}\} = -\cosh(z(H_1 - H_2)) F_{32} && \{F_{23}, F_{31}\} = \cosh(z(H_2 - H_3)) F_{21}
\end{aligned}
$$

$$
\{F_{13}, F_{31}\} = \frac{\sinh(z(H_1 - H_3))}{z} + z \sinh(z(H_1 - H_2)) F_{23} F_{32} + z \sinh(z(H_2 - H_3)) F_{12} F_{21}.
$$

The fact that $\Delta_z^{\mathcal{P}}$ is a Poisson algebra homomorphism between $su_q^{\mathcal{P}}(3)$ and $su_q^{\mathcal{P}}(3) \otimes su_q^{\mathcal{P}}(3)$ can be proven by direct computation.

4.3. Casimir functions

In the case of the quantum algebra $U_{z,\hbar}(su(3))$ the problem of finding its q-deformed Casimir operators is a quite difficult one (see for instance the construction for the $U_q(sl(3))$ ones given in [31]). Nevertheless, the corresponding central functions in the case of $su_q^{\mathcal{P}}(3)$ can be obtained by direct computation. The explicit form of the deformed 'second order' Casimir function is

$$
\begin{aligned}
C_z^{\mathcal{P}} = \frac{1}{z^2} \Bigg(&\sinh^2 \frac{z(H_1 + H_2 - 2H_3)}{3} + \sinh^2 \frac{z(H_1 + H_3 - 2H_2)}{3} + \sinh^2 \frac{z(H_2 + H_3 - 2H_1)}{3} \Bigg)\\
&+ 2F_{12}F_{21} \cosh \frac{z(H_1 + H_2 - 2H_3)}{3} + 2F_{23}F_{32} \cosh \frac{z(H_2 + H_3 - 2H_1)}{3}\\
&+ 2F_{31}F_{13} \cosh \frac{z(H_3 + H_1 - 2H_2)}{3} + 2z(F_{12}F_{23}F_{31} + F_{21}F_{32}F_{13})\\
&\times \sinh \frac{z(H_3 + H_1 - 2H_2)}{3} + 2z^2 F_{12}F_{21}F_{32}F_{23} \cosh \frac{z(H_3 + H_1 - 2H_2)}{3}.
\end{aligned}
$$

The 'third order' one reads

$$\begin{aligned} D_z^{\mathcal{P}} = &-\frac{1}{z^3}\sinh\frac{z(H_1+H_2-2H_3)}{3}\sinh\frac{z(H_1+H_3-2H_2)}{3}\sinh\frac{z(H_2+H_3-2H_1)}{3}\\ &+\frac{1}{z}F_{12}F_{21}\sinh\frac{z(H_1+H_2-2H_3)}{3}+\frac{1}{z}F_{23}F_{32}\sinh\frac{z(H_2+H_3-2H_1)}{3}\\ &+\frac{1}{z}F_{31}F_{13}\sinh\frac{z(H_3+H_1-2H_2)}{3}+(F_{12}F_{23}F_{31}+F_{21}F_{32}F_{13})\\ &\times\cosh\frac{z(H_3+H_1-2H_2)}{3}+zF_{12}F_{21}F_{32}F_{23}\sinh\frac{z(H_3+H_1-2H_2)}{3}. \end{aligned}$$

These two expressions can be useful in two different directions. First, as the building blocks for the construction of integrable deformations of $su(3)$ classical spin chains through the formalism given in [22]. Second, as a first-order guide to construct the q-deformed Casimir operators for the quantum algebra $U_{z,\hbar}(su(3))$ having in mind the analysis of the corresponding q-deformed mass formulae [32].

5. Conclusions

The algebraic structures presented in this paper illustrate the full 'hierarchy of complexity' that Hopf algebras provide: quantum algebras $U_{z,\hbar}(g)$ would be the richest and most complex structures (both non-commutative since $\hbar \neq 0$ and non-co-commutative since $z \neq 0$), the q-Poisson algebras would be somehow intermediate (being commutative and non-co-commutative) and the Lie algebras would be the 'simplest' ones (non-commutative and co-commutative).

In this context, the procedure to obtain a q-Poisson analogue of a given quantum algebra is canonically defined. First, one goes from $U_z(g)$ to $U_{z,\hbar}(g)$ by constructing analytically the deformation from the very beginning and by taking into account explicitly both parameters $(\hbar, z)$. At first sight, this two-parameter deformation seems to be irrelevant since for any finite value of $\hbar$, by using the Lie algebra automorphism $X \to \hbar X$ and by transforming the deformation parameter as $z \to z/\hbar$, the quantum algebra $(U_{z,\hbar}(g), \Delta_{z,\hbar})$ can be converted into $(U_z(g), \Delta_z)$. However, in order to obtain the Poisson analogue, one has to perform the $\hbar \to 0$ limit that *does not commute* with this automorphism and gives rise to the right result. The approach has been discussed in all details for the coproduct, the extension to co-unity and antipode is trivial. Note that this procedure is quite similar to the well-known contraction theory of quantum algebras [33].

With respect to the Poisson limit (4) and (5) we would like to stress that it is essential to write the quantum algebra in terms of commutation rules, and *not* by making use of q-commutators, since the Poisson limit of the latter is not well defined. This fact did not allow in the past the construction of q-Poisson analogues for algebras of rank greater than 1. However, the analytical bases approach presented in [24, 25] provides a quantization framework based on pure commutators, thus leading to a well-defined Poisson limit for arbitrary quantum algebras. Hence, analytical bases look also to have a privileged connection with the semiclassical limit.

On the other hand, quantum deformations of simple Lie algebras are usually described by means of their simple root generators X_i together with their q-Serre relations. Indeed, this approach simplifies the mathematical scheme as it does not make use of the non-simple root generators, even if the latter are necessary in the physical applications due to their role as independent symmetries (see, for instance, [11, 15]). However, the q-Serre relations can be rewritten in terms of commutators (see [29, 30]) and, after introducing explicitly the $\hbar$ parameter they read

$$\begin{aligned} &[X_i, [X_i, X_j]] = 4\sinh^2\frac{z\hbar}{2} X_i X_j X_i && \text{if} \quad a_{ij} = -1, \\ &[X_i, X_j] = 0 && \text{if} \quad a_{ij} = 0. \end{aligned} \tag{22}$$

Therefore, the corresponding q-Poisson–Serre relations in the limit $\hbar \to 0$ can be consistently defined as

$$\begin{aligned} &\{X_i, \{X_i, X_j\}\} = z^2 X_i^2 X_j && \text{if} \quad a_{ij} = -1, \\ &\{X_i, X_j\} = 0 && \text{if} \quad a_{ij} = 0. \end{aligned} \tag{23}$$

Acknowledgments

This work was partially supported by the Ministerio de Educación y Ciencia of Spain (Projects FIS2005-03989, FIS2009-09002 and MTM2007-67389, all of them with EU-FEDER support), by the Junta de Castilla y León (Project GR224), and by INFN-MICINN (Italy-Spain).

References

[1] Kulish P P and Reshetikhin N 1981 *Proc. Steklov Math. Inst.* **101** 101
[2] Takhtajan L A 1990 *Introduction to Quantum Group and Integrable Massive Models of Quantum Field Theory* ed Ge Mo-Lin and Bao-Heng Zao (Singapore: World Scientific) p 69
[3] Reshetikhin N Y, Takhtadzhyan L A and Faddeev L D 1990 *Leningrad Math. J.* **1** 193
[4] Chari V and Pressley A 1994 *A Guide to Quantum Groups* (Cambridge: Cambridge University Press)
[5] Majid S 1995 *Foundations of Quantum Group Theory* (Cambridge: Cambridge University Press)
[6] Pasquier V and Saleur H 1990 *Nucl. Phys.* B **330** 523
[7] Kulish P P and Sklyanin E K 1991 *J. Phys. A: Math. Gen.* **24** L435
[8] Gómez C, Ruiz-Altaba M and Sierra G 1995 *Quantum Groups in Two-Dimensional Physics* (Cambridge: Cambridge University Press)
[9] Bonechi F, Celeghini E, Giachetti R, Sorace E and Tarlini M 1992 *Phys. Rev. Lett.* **68** 3718
[10] Bonechi F, Celeghini E, Giachetti R, Sorace E and Tarlini M 1992 *Phys. Rev.* B **46** 5727
[11] Lukierski J, Nowicky A and Ruegg H 1992 *Phys. Lett.* B **293** 344
[12] Giller S, Koshinski P, Majewski M, Maslanka P and Kunz J 1992 *Phys. Lett.* B **286** 57
[13] Ruegg H 1993 *Integrable Systems, Quantum Groups and Quantum Field Theories (NATO ASI Series)* ed L A Ibort and M A Rodríguez (The Netherlands: Kluwer) p 45
[14] Dobrev V K 1993 *J. Phys. A: Math. Gen.* **26** 1317
[15] Ballesteros A, Herranz F J, del Olmo M A and Santander M 1994 *J. Math. Phys.* **35** 4928
[16] Celeghini E, Giachetti R, Sorace E and Tarlini M 1992 *Phys. Lett.* B **280** 180
[17] Bonatsos D and Daskaloyannis C 1999 *Prog. Part. Nucl. Phys.* **43** 537
[18] Gaudin M 1998 *J. Physique* **37** 1087
[19] Sklyanin E K 1992 *Commun. Math. Phys.* **150** 181
[20] Falqui G and Musso F 2003 *J. Phys. A: Math. Gen.* **36** 11655
[21] Lerma H S and Errea B F 2007 *J. Phys. A: Math. Theor.* **40** 4125
[22] Ballesteros A and Ragnisco O 1998 *J. Phys. A: Math. Gen.* **31** 3791
[23] Da Providencia J 1993 *J. Phys. A: Math. Gen.* **26** 5845
[24] Ballesteros A, Celeghini E and del Olmo M A 2008 *J. Phys. C. S.* **128** 012049
[25] Celeghini E, Ballesteros A and del Olmo M A 2008 *J. Phys. A: Math. Theor.* **41** 304038
[26] Drinfeld V G 1987 'Quantum groups' *Proc. Int. Congress of Mathematicians Berkeley, 1986* ed A M Gleason (Providence: AMS)
[27] Jimbo M 1985 *Lett. Math. Phys.* **10** 63
[28] Ohn C 1992 *Lett. Math. Phys.* **25** 85
[29] Ballesteros A, Celeghini E and del Olmo M A 2006 *J. Phys. A: Math. Gen.* **39** 9161
[30] Ballesteros A, Celeghini E and del Olmo M A 2007 *J. Phys. A: Math. Theor.* **40** 2013
[31] Rodríguez-Plaza M J 1991 *J. Math. Phys.* **32** 2020
[32] Cárcamo Hernández A E 2006 *J. Math. Phys.* **47** 073509
[33] Celeghini E, Giachetti R, Sorace E and Tarlini M 1992 *Lecture Notes in Mathematics* vol 1510 ed P P Kulish (Berlin: Springer) p 221